Relativistic dynamics of accelerating particles derived from field equations.

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March 12, 2012

Abstract

In relativistic mechanics the energy-momentum of a free point mass moving without acceleration forms a four-vector. Einstein's celebrated energy-mass relation $E = mc^2$ is commonly derived from that fact. By contrast, in Newtonian mechanics the mass is introduced for an accelerated motion as a measure of inertia. In this paper we rigorously derive the relativistic point mechanics and Einstein's energy-mass relation using our recently introduced neoclassical field theory where a charge is not a point but a distribution. We show that both the approaches to the definition of mass are complementary within the framework of our field theory. This theory also predicts a small difference between the electron rest mass relevant to the Penning trap experiments and its mass relevant to spectroscopic measurements.

1 Introduction

The concept of a point particle is at the very heart of Newtonian mechanics. Regarding this, Einstein said, "Physical events, in Newton's view, are to be regarded as the motions, governed by fixed laws, of material points in space. The material point is our only mode of representing reality when dealing with changes taking place in it" [EinIO]. Even in the quantum mechanics the concept of a point particle continues to be fundamentally exact. As Feynman puts it, "The wave function $\psi(\mathbf{r})$ for an electron in an atom does not, then, describe a smeared-out electron with a smooth charge density. The electron is either here, or there, or somewhere else, but wherever it is, it is a point charge" [FeyIII]. One of our key motivations for introducing a neoclassical field theory of distributed elementary charges in [BF4]-[BF7] was a desire to account for particle properties as well as for wave phenomena in a single mathematically sound Lagrangian relativistic theory. This theory is self-contained and, consequently, all particle properties must naturally come out from the field equations as approximations. We have already demonstrated that the theory implies in the non-relativistic limit (i) the non-relativistic particle mechanics governed by the Newton equations with the Lorentz forces and (ii) the frequency spectrum for hydrogenic atoms. In this paper we study relativistic aspects of the theory, namely, the motion of a localized wave which describes the distributed charge in an external EM field. An idea of a particle emerging as a well localized field was explored by a number of authors, in particular in the form of "extended charge" models. The Lorentz-Abraham model and its developments was studied and advanced in

[AppKie], [BamGal], [IKM], [Jack], [Kie2], [KKSpo], [Nod], [Pea1], [Roh], [Schwin], [Spo], [Yag]. In 1905-1906 Poincaré suggested [Poi] (see also [Jack], [Roh], [PauRT], [Schwin], [Yag] and references within) adding non-electromagnetic cohesive forces to the Lorentz-Abraham model. Here we study dynamics in an external field of a distributed charge with such cohesive forces.

Recall now the fundamentals of the relativistic dynamics of a mass point and the relativistic field theory. The relativistic dynamics of an accelerating mass point charge in the case where the acceleration is caused by the electromagnetic (EM) external field $\mathbf{E}_{\text{ex}}$, $\mathbf{B}_{\text{ex}}$ is described (see, for example, [Barut], [PauRT]) by the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\mathbf{v}\right)=\mathbf{f}_{\text{Lor}}\left(t,\mathbf{r}\right),\quad M=m_0\gamma,\quad \gamma=\left(1-\frac{\mathbf{v}^2}{\mathrm{c}^2}\right)^{-1/2}, \tag{1}$$

where $\mathbf{v}=d\mathbf{r}/dt$ is its velocity, m_0 is the rest mass of the mass point, $\mathbf{f}_{\text{Lor}}$ is the Lorentz force,

$$\mathbf{f}_{\text{Lor}}\left(t,\mathbf{r}\right)=q\left(\mathbf{E}_{\text{ex}}\left(t,\mathbf{r}\right)+\frac{1}{\mathrm{c}}\mathbf{v}\times\mathbf{B}_{\text{ex}}\left(t,\mathbf{r}\right)\right), \tag{2}$$

q is its charge and γ is known as the Lorentz factor. Equations (1) for the space components of the 4-vector are usually complemented with the time component

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M\mathrm{c}^2\right)=\mathbf{f}_{\text{Lor}}\cdot\mathbf{v}. \tag{3}$$

In particular, for small velocities when $\left|\mathbf{v}\right|/\mathrm{c}\ll 1$ we readily recover from equation (1) as its non-relativistic approximation Newton's equation with the Lorentz force by setting $\gamma=1$. Note that in (1) the rest mass m_0 of a point is an intrinsic property of a point and is prescribed.

In a relativistic field theory the relativistic field dynamics is derived from a relativistic covariant Lagrangian. The field equations, total energy, momentum, forces and their densities are naturally defined in terms of the Lagrangian for both closed and non closed (with external forces) systems (see, for instance, [And], [Barut], [Mol], [PauRT], [Sexl], see also Section 2.1). For a closed system the total momentum has a simple form $\mathbf{P}=M\mathbf{v}$ with a constant velocity $\mathbf{v}$, and the relativistic invariance of the energy-momentum 4-vector allows to derive Einstein's celebrated energy-mass relation,

$$\mathsf{E}=M\mathrm{c}^2\quad M=m_0\gamma, \tag{4}$$

between the mass M and the energy E (see, for instance, [And, Sec. 7.1-7.5], [Mol, Sec. 3.1-3.3, 3.5], [PauRT, Sec. 37], [Sexl, Sec. 4.1]). As stated by Pauli, [PauRT, p. 123]: "We can thus consider it as proved that the relativity principle, in conjunction with the momentum and energy conservation laws, leads to the fundamental principle of the equivalence of mass and (any kind of) energy. We may consider this principle (as was done by Einstein) as the most important of the results of the theory of special relativity." Yet another important point about the Einstein mass-energy relations is made by Laue [Schilpp, p. 529]: "... we can determine the total amount of energy in a body from its mass. We thereby get rid of the arbitrariness of the zero point of energy which the former definition of energy (cf. Section III) was forced to introduce. There are not merely energy differences, as before; the energy possesses a physically meaningful absolute value."

Now let us take a brief look at the difference in determining the mass in the dynamics of a single point versus a general Lagrangian Lorentz invariant field theory.

First, since equation (1) for a single mass point has a form of Newton's law, one can determine as in Newtonian mechanics the mass M as a measure of inertia from the known force $\mathbf{f}_{\text{Lor}}$ and acceleration $\partial_t^2 \mathbf{r}$ (the variability of γ can be ignored for mild accelerations). In a relativistic field theory for a closed system the energy-momentum is a four-vector, and that allows to define the total mass and the rest mass of the system in terms of the energy by Einstein's formula (4) in the case of uniform motion. However, in the case of a general non-closed system (which is the subject of our primary interest since we study field regimes with acceleration), there is no canonical way to determine the mass, position, velocity, and acceleration. For a non-closed system there is even a problem with a sound definition of the center of mass since "the centre of mass loses its physical importance" [Mol, p. 203]. To summarize, in a general relativistic field theory the rest mass is defined for a uniform motion, whereas in the Newtonian mechanics the concept of inertial mass is introduced through an accelerated motion.

Our manifestly relativistic Lagrangian field theory describes a single charge by a complex-valued scalar field (wave function) $\psi(t, \mathbf{x})$ satisfying the nonlinear Klein-Gordon (KG) equation,

$$-\frac{1}{\mathrm{c}^2}\tilde{\partial}_t^2\psi + \tilde{\nabla}^2\psi - G'\left(|\psi|^2\right)\psi - \frac{m^2\mathrm{c}^2}{\chi^2}\psi = 0, \tag{5}$$

where m is a positive mass parameter, and χ is a constant which coincides with (or is close to) the Planck constant $\hbar$. The expressions for the covariant derivatives in (5) are

$$\tilde{\partial}_t = \partial_t + \frac{\mathrm{i}q}{\chi}\varphi_{\text{ex}}, \quad \tilde{\nabla} = \nabla - \frac{\mathrm{i}q}{\chi\mathrm{c}}\mathbf{A}_{\text{ex}}, \tag{6}$$

where q is the value of the charge, and $\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}$ are the potentials of the *external EM field* (which can be thought of as the one produced by all remaining charges of the original system of many charges, see [BF4]-[BF6] for details). The nonlinear term G' is given by the logarithmic expression

$$G'(s) = G'_a(s) = -a^{-2}\left[\ln\left(a^3 s\right) + \ln \pi^{3/2} + 3\right], \quad s \geq 0, \tag{7}$$

where a is the *charge size parameter*. It is established below that for certain regimes of accelerated motion the relativistic mass point equations (1)-(3) are an approximation which describes the behavior of the field ψ when it is well localized. The charge localization is facilitated by the Poincaré-like cohesive forces associated with the nonlinearity $G'_a(s)$. In particular, a free resting charge with the minimal energy and size a has a Gaussian shape, namely, $|\psi| = \pi^{-3/4}a^{-3/2}e^{-|x|^2a^{-2}/2}$. The localization can be described by the ratio a/R_f where R_f is a typical length scale of the spatial variation of the external EM forces.

As it is commonly done, we start with assigning a position to the distributed charge by using its energy density $u(t, \mathbf{x})$ to define its total energy E and *energy center (or ergocenter)* $\mathbf{r}(t)$ by the formulas

$$\mathsf{E}(t) = \int u(t, \mathbf{x})\, \mathrm{d}\mathbf{x}, \quad \mathbf{r}(t) = \frac{1}{\mathsf{E}(t)}\int \mathbf{x}u(t, \mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{8}$$

If a field ψ satisfies the field equation (5), its ergocenter $\mathbf{r}$ and energy E satisfy equations which can be derived from the conservation laws for the KG equation (5). We prove under the assumption of localization in the asymptotic limit $a/R_f \to 0$ that these equations turn

into the relativistic mass point equations (1), (3). Remarkably, the value of the inertial mass determined from the equations by the Newtonian approach coincides exactly with the mass given by Einstein's formula (4). Of course, a convincing argument for the equivalence of the inertial mass and the energy based on the analysis of the charge momentum when it interacts with the electromagnetic field has been made by Einstein [Ein05a], but here the same is obtained through a thorough mathematical analysis of a concrete Lagrangian model.

Since our Lagrangian theory is self-contained, Einstein's energy-mass formula (4) or any asymptotic law of motion must be derived within the framework of the theory. *Importantly, since the limit mass point equations are derived, and not postulated, the resulting rest mass is shown to be an integral of motion rather than a prescribed constant. Consequently, the rest mass may take different values depending on the state of the field. In particular, in addition to the primary Gaussian ground state there is a sequence of rest states with higher rest energies and rest masses.* The possibility of different rest masses comes from the fact that in our theory an elementary charge is not a point but is a distribution described by a wave function ψ. The charge although elementary has infinitely many degrees of freedom, with internal interactions of not electromagnetic origin, contributing to its internal energy.

Our theory yields a simple expression for the rest mass m_0 of a charge which differs slightly from the mass parameter m, namely,

$$m_0 = m + \frac{m}{2}\frac{a_{\rm C}^2}{a^2}, \tag{9}$$

where $a_{\rm C}$ is the reduced Compton wavelength. Evidently, the difference $m_0 - m$ vanishes as $a_{\rm C}/a \to 0$. In the nonrelativistic case treated in [BF4]-[BF6] the nonlinear Schrodinger equation follows from the KG equation in the limit $\mathrm{c} \to \infty$ and $\alpha \to 0$ where $\alpha = q^2\chi^{-1}\mathrm{c}^{-1}$ is the Sommerfeld fine structure constant. In this limit the mass parameter m coincides with the inertial mass m_0 in evident agreement with the limit of relation (9) with a fixed Bohr radius $a_{\rm B} = a_{\rm C}/\alpha$. Observe that our theory provides for complementary interpretations of the relativistic and non-relativistic masses which are sometimes considered (see [EriVoy]) to be "rival and contradictory."

The focus of this paper is on the charge motion in an external EM field which has a mild spatial variation. The external EM fields with stronger spatial variation are covered by the theory as well but with the use of other techniques. Namely, in the case of the hydrogen atom [BF6], [BF7], for the electron in the Coulomb field we derive the well known energy spectrum involving the Rydberg constant R_∞ with the mass parameter m in the place of the electron mass. The difference between the mass m entering the spectroscopic data and the inertial mass m_0 as it appears in the Penning trap experiments is discussed in Section 2.5.

In the following sections we analyze relativistic features of the charge emerging from the underlying field dynamics. In particular, in Section 2 we derive the mass point equations (1), (3) under an assumption that the charge wave function remains localized. In Section 3 we show that the localization assumption is consistent with the KG equation (5). In particular, we describe there a class of external EM fields in which the charge maintains its localization in the strongest possible form when it accelerates according to the KG field equation (5). The relativistic theory of many interacting charges is left for another paper.

2 Relativistic distributed charge as a particle

In [BF4]-[BF7] we developed a neoclassical theory for many interacting charges based on a relativistic and gauge invariant Lagrangian. We have demonstrated there that this Lagrangian theory describes electromagnetic interaction in all spatial scales: it accounts for at least some quantum phenomena at atomic scale including the frequency spectrum of the hydrogen atom, and it accounts for the classical motion of non-relativistic charges when they are well separated and localized. In this paper we study the relativistic aspects of our theory in the case of a single charge. Our general Lagrangian in this special case turns into the following relativistic and gauge invariant expression

$$\mathcal{L}_1(\psi) = \frac{\chi^2}{2m}\left\{\frac{1}{\mathrm{c}^2}\tilde{\partial}_t\psi\tilde{\partial}_t^*\psi^* - \tilde{\nabla}\psi\cdot\tilde{\nabla}^*\psi^* - \kappa_0^2\psi^*\psi - G(\psi^*\psi)\right\}, \tag{10}$$

where $\psi(t,\mathbf{x})$ is a complex valued wave function over the space-time continuum and ψ^* is its complex conjugate. In the expression (10) c is the speed of light,

$$\kappa_0 = \frac{mc}{\chi}. \tag{11}$$

The covariant derivatives in (10) are defined by (6) and the nonlinearity $G(s)$ is defined by the formula

$$G(s) = G_a(s) = -a^{-2}s\left[\ln\left(a^3 s\right) + \ln\pi^{3/2} + 2\right], \quad s \geq 0, \tag{12}$$

where $a > 0$ is the size parameter. The field equation corresponding to the Lagrangian (10) is the nonlinear Klein-Gordon (KG) equation (5).

It is proven in [BF4]-[BF6] that in the *non-relativistic* case the trajectory of the ergocenter converges to a corresponding solution of Newton's equation with the Lorentz force as $a/R_f \to 0$ where R_f is a typical variation scale of the external EM field. It becomes our goal now *to derive from the KG field equation (5) the relativistic law of motion* for the ergocenter $\mathbf{r}(t)$. Namely, we show below that the ergocenter trajectory $\mathbf{r}(t)$ converges to a solution of the relativistic equation (1) as $a/R \to 0$ if the charge is localized. The derivation is based entirely on the analysis of the KG equation (5) and corresponding conservation laws.

2.1 Field symmetric energy-momentum tensor and conservation laws

There are a few popular conventions in setting up coordinates and the metric in the Minkowski four dimensional space-time. We pick the one which seems to be dominant nowadays as in [Barut, Section 1], [LanLif F, Sections 1.1-1.4, 2], [ItzZub, Section 1-1-1]. The time-space four-vector in its contravariant x^μ and covariant x_μ forms is represented as follows

$$x = x^\mu = \left(x^0, x^1, x^2, x^3\right) = (\mathrm{c}t, \mathbf{x}), \ \mu = 0,1,2,3; \tag{13}$$

$$x_\mu = g_{\mu\nu}x^\nu = \left(x^0, -x^1, -x^2, -x^3\right); \tag{14}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{\mathrm{c}}\partial_t, \nabla\right); \ \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{\mathrm{c}}\partial_t, -\nabla\right); \tag{15}$$

with the common convention on the summation of the same indices. The metric tensor $g_{\mu\nu} = g^{\mu\nu}$ is defined by

$$\{g_{\mu\nu}\} = \{g^{\mu\nu}\} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{16}$$

We also use a common convention for the space 3-vector

$$x^i = \left(x^1, x^2, x^3\right) = \mathbf{x},\ i = 1, 2, 3, \tag{17}$$

emphasizing notationally by the Latin superscript its difference from 4-vector x^μ with the Greek superscript.

For vector field potentials $(\varphi, \mathbf{A})$ in the KG equation (5), we use standard relativistic notations with the four-vector potential A^μ, four-vector current density J^ν, and the electromagnetic field $F^{\mu\nu}$:

$$A^\mu = (\varphi_{\text{ex}}, \mathbf{A}_{\text{ex}}), \quad J^\mu = (\mathrm{c}\varrho, \mathbf{J}), \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{18}$$

so that

$$F^{\mu\nu}_{\text{ex}} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}, \tag{19}$$

where, as always,

$$\mathbf{E} = -\nabla\varphi_{\text{ex}} - \frac{1}{\mathrm{c}}\partial_t \mathbf{A}_{\text{ex}},\ \mathbf{B} = \nabla \times \mathbf{A}_{\text{ex}}. \tag{20}$$

The *Lorentz force density* of the external EM field $F^{\mu\nu}_{\text{ex}}$ acting on a 4-current J^μ is of the form

$$f^\mu = \frac{1}{\mathrm{c}} F^{\mu\nu} J_\nu = \left(\frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{E}, \rho\mathbf{E} + \frac{1}{\mathrm{c}}\mathbf{J}\times\mathbf{B}\right). \tag{21}$$

We turn now to the analysis of the Lagrangian (10). As a consequence of its gauge invariance, we obtain the conserved four-current $J^\nu = (\mathrm{c}\rho, \mathbf{J})$ where the charge density and the current are defined by the following expressions, [BF7],

$$\begin{aligned} \rho &= -\left(\frac{\chi q}{m\mathrm{c}^2}\,\mathrm{Im}\,\frac{\partial_t\psi}{\psi} + \frac{q^2}{m\mathrm{c}^2}\varphi_{\text{ex}}\right)|\psi|^2, \\ \mathbf{J} &= \left(\frac{\chi q}{m}\,\mathrm{Im}\,\frac{\nabla\psi}{\psi} - \frac{q^2}{m\mathrm{c}}\mathbf{A}_{\text{ex}}\right)|\psi|^2. \end{aligned} \tag{22}$$

They satisfy the continuity equation

$$\partial_t\rho + \nabla\cdot\mathbf{J} = 0 \tag{23}$$

implying the charge conservation

$$\int_{\mathbb{R}^3} \rho\,(t, \mathbf{x})\ \mathrm{d}\mathbf{x} = \bar{\rho} = q, \tag{24}$$

where we choose the constant $\bar{\rho}$ to be exactly the charge q as it arises in Coulomb's law, see [BF4]-[BF7] for details.

There are well understood approaches, due to Belinfante and Rosenfeld, [LanLif F, Sec. 32], [Barut, Sec. III.4], [LanVPM, Sec. 22], to constructing the symmetric energy-momentum tensor (EnMT) $T^{\mu\nu}$ from the Lagrangian $\mathcal{L}$ based on its invariance with respect to the Poincaré group. The EnMT $T^{\mu\nu}$ corresponding to our Lagrangian $\mathcal{L}_1$ is constructed by the same method as in [BF4] and [BF5] yielding as it does there

$$T^{\mu\nu} = \frac{\chi^2}{2m}\left\{\left[\psi^{;\mu *}\psi^{;\nu} + \psi^{;\mu}\psi^{;\nu *}\right] - \left[\psi^*_{;\mu}\psi^{;\mu} - \kappa_0^2\psi^*\psi - G\left(\psi^*\psi\right)\right]g^{\mu\nu}\right\}, \tag{25}$$

where the covariant derivatives $\psi^{;\mu}$ are defined by

$$\psi^{;\mu} = \left(\partial^\mu + \frac{\mathrm{i}qA^\mu_{\mathrm{ex}}}{\chi \mathrm{c}}\right)\psi = \left(\frac{1}{\mathrm{c}}\tilde{\partial}_t\psi, -\tilde{\nabla}\psi\right),\ \tilde{\partial}_t = \partial_t + \frac{\mathrm{i}q\varphi_{\mathrm{ex}}}{\chi},\ \tilde{\nabla} = \nabla - \frac{\mathrm{i}q\mathbf{A}_{\mathrm{ex}}}{\chi \mathrm{c}}, \tag{26}$$

and $\psi^{*;\mu}$ is the complex conjugate to $\psi^{;\mu}$.

We proceed with the interpretation of entries the symmetric EnMT $T^{\mu\nu}$, [LanLif F, Section 32], [MorFesh1, Chapter 3.4]

$$T^{\mu\nu} = T^{\mu\nu} = \begin{bmatrix} u & \mathrm{c}p^1 & \mathrm{c}p^2 & \mathrm{c}p^3 \\ \mathrm{c}^{-1}s^1 & -\sigma^{11} & -\sigma^{12} & -\sigma^{13} \\ \mathrm{c}^{-1}s^2 & -\sigma^{21} & -\sigma^{22} & -\sigma^{23} \\ \mathrm{c}^{-1}s^3 & -\sigma^{31} & -\sigma^{32} & -\sigma^{33} \end{bmatrix}, \quad \begin{array}{|l|l|} \hline u & \text{energy density,} \\ \hline p^j & \text{momentum density,} \\ \hline s^j = \mathrm{c}^2 p^j & \text{energy flux density,} \\ \hline \sigma^{ij} = \sigma^{ji} & \text{symmetric stress tensor.} \\ \hline \end{array}, \tag{27}$$

where $i, j = 1, 2, 3$.

Remark 1 *As a consequence of the symmetry of EnMT $T^{\mu\nu}$ we have the following relation between the field energy flux and the field momentum densities*

$$\mathbf{s} = \mathrm{c}^2\mathbf{p} \tag{28}$$

W. Pauli refers to the identity (28) as a theorem and makes a comment similar to that of C. Lanczos, [PauRT, p. 125]: "This is the theorem of the momentum of the energy current, first expressed by Planck[229] according to which a momentum is associated with each energy current. This theorem can be considered as an extended version of the principle of the equivalence of mass and energy. Whereas the principle only refers to the total energy, the theorem has also something to say on the localization of momentum and energy."There is an intimate relation between the concept of particle and the field concept of the symmetric energy-momentum tensor *(EnMT) $T^{\mu\nu}$. In particular, the fundamental Einstein mass-energy relation $\mathsf{E}_0 = m_0\mathrm{c}^2$ can be interpreted as the symmetry of the energy-momentum tensor, a point stressed by C. Lanczos, [LanVPM, p. 394]: " It was Planck in 1909 who pointed out that the field theoretical interpretation of Einstein's principle can only be the symmetry of the energy-momentum tensor. If the T_{i4} ($i = 1, 2, 3$) (i.e. the momentum density) and the T_{4i}, the energy current, did not agree, then the conservation of mass and energy would follow different laws and the principle $m = E$ could not be maintained. Nor could a non-symmetric energy-momentum tensor guarantee the law of inertia, according to which the centre of mass of an isolated system moves in a straight line with constant velocity."*

The Noether theorem then implies 10 conservation laws: the *energy and momentum conservation laws*

$$\partial_\mu T^{\mu\nu} = f^\nu, \quad f^\nu = -\frac{\partial\mathcal{L}}{\partial x_\nu}, \tag{29}$$

where f^ν is the force density, and the *angular momentum conservation law*, [Sexl, Sec. 10.2], [Barut, Sec. II.1, III.4],

$$\partial_\mu M^{\mu\nu\gamma} = N^{\nu\gamma}, \text{ where} \tag{30}$$
$$M^{\mu\nu\gamma} = x^\nu T^{\mu\gamma} - x^\gamma T^{\mu\nu}, \quad N^{\nu\gamma} = x^\nu f^\gamma - f^\nu x^\gamma,$$

$M^{\mu\nu\gamma}$ is the *angular momentum density tensor,* and $N^{\nu\gamma}$ is the *torque density tensor*. Notice that the angular momentum conservation follows readily from the energy and momentum conservation law (29) combined with the symmetry of the EnMT $T^{\mu\nu}$.

Using the interpretation (27) we can recast the energy-momentum conservation laws (29) in a more appealing form

$$\partial_t p^i = \sum_{j=1,2,3} \partial_j \sigma^{ji} - f^i, \quad f^i = \frac{\partial \mathcal{L}}{\partial x_i}, \ i = 1,2,3, \tag{31}$$

$$\partial_t u = -\sum_{j=1,2,3} \partial_j s^j - f^0, \quad f^0 = -\frac{\partial \mathcal{L}}{\partial t}. \tag{32}$$

In the case of a closed system when $f^i = 0$, the total conserved quantities are, [Barut, (3.76)-(3.77)], [PauRFT, (6), (12), (15b)], [Sexl, Sec. 10.2],

$$P^\nu = \int_\sigma T^{\mu\nu}\, \mathrm{d}\sigma_\mu, \ J^{\nu\gamma} = \int_\sigma M^{\mu\nu\gamma} \mathrm{d}\sigma_\mu, \tag{33}$$

where σ is any space-like surface, for instance $x_0 = \text{const}$. P^ν is *the four-vector of the total energy-momentum* and $J^{\nu\gamma} = -J^{\gamma\nu}$ is *the total angular momentum tensor*. Importantly, *for closed systems the conserved total energy-momentum P^ν and angular momentum $J^{\nu\gamma}$ transform respectively as 4-vector and 4-tensor* under Lorentz transformation, and that is directly related to the conservations laws, [Mol, Section 6.2], [Jack, Section 12.10 A]. But *for open (not closed) systems generally the total energy-momentum P^ν and $J^{\nu\gamma}$ angular momentum do not transform as respectively 4-vector and 4-tensor*, [Mol, Section 7.1, 7.2], [Jack, Section 12.10 A, 16.4].

The formula for the 4-microcurrent density J^μ turns into

$$J^\nu = -\frac{\chi q}{2m} \mathrm{i} \left(\psi \tilde{\partial}^{\nu *} \psi^* - \psi^* \tilde{\partial}^\nu \psi \right).$$

Using the same method as in [BF4, Sec. 11.7] we find that for the Lagrangian (10) the general conservation laws (29) turn into

$$\partial_\mu T^{\mu\nu} = f^\nu, \tag{34}$$

where

$$f^\nu = \frac{1}{\mathrm{c}} J_\mu F^{\nu\mu}_{\text{ex}} = \left(f^0, \mathbf{f}\right) = \left(\frac{1}{\mathrm{c}} \mathbf{J} \cdot \mathbf{E}_{\text{ex}}, \rho \mathbf{E}_{\text{ex}} + \frac{1}{\mathrm{c}} \mathbf{J} \times \mathbf{B}_{\text{ex}} \right) \tag{35}$$

with $\mathbf{J}$,ρ given in (22). Evidently, $\mathbf{f}$ is the Lorentz force density for the external field $F^{\nu\mu}_{\text{ex}}$. Using the interpretation of the EnMT entries (28) and (25) we get the following representations for the energy and the momentum densities:

$$u = \frac{\chi^2}{2m} \left[\frac{1}{\mathrm{c}^2} \tilde{\partial}_t \psi \tilde{\partial}_t^* \psi^* + \tilde{\nabla} \psi \tilde{\nabla}^* \psi^* + G\left(\psi^* \psi\right) + \kappa_0^2 \psi \psi^* \right], \tag{36}$$

$$\mathbf{p} = \left(p^1, p^2, p^3\right) = -\frac{\chi^2}{2m\mathrm{c}^2} \left(\tilde{\partial}_t \psi \tilde{\nabla}^* \psi^* + \tilde{\partial}_t^* \psi^* \tilde{\nabla} \psi \right). \tag{37}$$

2.2 Relativistic dynamics of the energy center of a localized charge

Here we derive equations for the energy center $\mathbf{r}$ based on the conservation laws introduced in the previous section. Then based on these equations under asymptotic localization assumptions we derive the relativistic point mass equations (1), (3) for the energy center $\mathbf{r}$. We set first the total energy, momentum and the force expressions respectively by

$$\mathsf{E}(t)=\int u(t,\mathbf{x})\,\mathrm{d}\mathbf{x},\quad P^{i}(t)=\int p^{i}(t,\mathbf{x})\,\mathrm{d}\mathbf{x},\quad F^{i}(t)=\int f^{i}(t,\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{38}$$

The coordinates r^i of the energy center $\mathbf{r}$ are given then by

$$r^{i}=\frac{1}{\mathsf{E}(t)}\int x^{i}u(t,\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{39}$$

The continuity equation (23) when multiplied by $(\mathbf{x}-\mathbf{r})^i$ readily implies the following expression for the current density $\mathbf{J}$:

$$\partial_t\left((\mathbf{x}-\mathbf{r})^{i}\rho\right)+\rho\partial_t\mathbf{r}^{i}+\nabla\cdot\left((\mathbf{x}-\mathbf{r})^{i}\mathbf{J}\right)=\mathbf{J}^{i}. \tag{40}$$

Integrating over the entire space the energy-momentum conservation laws (31) and (32) we get

$$\frac{1}{\mathrm{c}}\partial_t\mathsf{E}=F^{0},\quad \partial_t P^{i}=F^{i}. \tag{41}$$

For the energy component in (41) using (40) we obtain

$$F^{0}=\int\frac{1}{\mathrm{c}}\mathbf{J}\cdot\mathbf{E}_{\mathrm{ex}}\,\mathrm{d}\mathbf{x}=\bar{\rho}\frac{1}{\mathrm{c}}\partial_t\mathbf{r}\cdot\mathbf{E}_{\mathrm{ex}}(t,\mathbf{r})+\delta_F^{0}, \tag{42}$$

$$\delta_F^{0}=\frac{1}{\mathrm{c}}\sum_i\int\left(\mathbf{E}_{\mathrm{ex}}^{i}\partial_t\left((\mathbf{x}-\mathbf{r})^{i}\rho\right)+\mathbf{E}_{\mathrm{ex}}^{i}\nabla\cdot\left((\mathbf{x}-\mathbf{r})^{i}\mathbf{J}\right)\right)\mathrm{d}\mathbf{x}+ \tag{43}$$

$$+\frac{1}{\mathrm{c}}\partial_t\mathbf{r}\cdot\int\rho\left(\mathbf{E}_{\mathrm{ex}}(t,\mathbf{x})-\mathbf{E}_{\mathrm{ex}}(t,\mathbf{r})\right)\mathrm{d}\mathbf{x}.$$

Integrating the angular momentum conservation law (30) for $\nu=0$ and $\gamma=i$ over the entire space we obtain

$$\frac{1}{\mathrm{c}}\partial_t\left(\mathrm{c}^2tP^{i}-r^{i}\mathsf{E}\right)=\mathrm{c}tF^{i}-F^{0}r^{i}-\delta_f^{i} \tag{44}$$

where the *particle discrepancy terms* δ_f^i are of the form

$$\delta_f^{i}=\int\left(x^{i}-r^{i}\right)f^{0}\,\mathrm{d}\mathbf{x}. \tag{45}$$

The identities (41) and (44) imply that

$$P^{i}=\frac{\mathsf{E}}{\mathrm{c}^2}\partial_t r^{i}-\frac{1}{\mathrm{c}}\delta_f^{i},\quad i=1,2,3. \tag{46}$$

Now, combining (46) with the second equality in (41), we obtain the relation

$$\partial_t\left(\frac{1}{\mathrm{c}^2}\mathsf{E}\partial_t\mathbf{r}-\frac{1}{\mathrm{c}}\boldsymbol{\delta}_f\right)=\mathbf{F},\quad ,i=1,2,3, \tag{47}$$

with $\mathbf{F} = (F^1, F^2, F^3)$ defined by (38), (35). The expressions for $\mathbf{F}$ can be written in the form

$$\mathbf{F} = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}) + \boldsymbol{\delta}_F, \tag{48}$$

where $\mathbf{f}_{\text{Lor}}(t, \mathbf{r})$ is the Lorentz force (2) with the remainder term

$$\begin{aligned} \boldsymbol{\delta}_F = \int (\mathbf{E}_{\text{ex}}(t, x) - \mathbf{E}_{\text{ex}}(t, r) + \mathrm{c}^{-1}\partial_t \mathbf{r}^i \times (\mathbf{B}_{\text{ex}}(x) - \mathbf{B}_{\text{ex}}(r)))\rho \mathrm{d}\mathbf{x} + \\ + \int \frac{1}{\mathrm{c}} \left(\partial_t ((\mathbf{x} - \mathbf{r})\rho) + \sum_l \partial_l \left((\mathbf{x} - \mathbf{r}) \mathbf{J}^l \right) \right) \times \mathbf{B}_{\text{ex}} \mathrm{d}\mathbf{x}. \end{aligned} \tag{49}$$

Equations (42) and (47) result in the following system of two ergocenter equations: the spatial part

$$\partial_t \left(\frac{\mathsf{E}}{\mathrm{c}^2} \partial_t \mathbf{r} - \boldsymbol{\delta}_f \right) = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}) + \boldsymbol{\delta}_F, \tag{50}$$

and the time part

$$\frac{1}{\mathrm{c}} \partial_t \mathsf{E} = \frac{1}{\mathrm{c}} \partial_t \mathbf{r} \cdot \mathbf{f}_{\text{Lor}}(t, \mathbf{r}) + \delta_F^0. \tag{51}$$

Observe now that the particle discrepancy terms $\boldsymbol{\delta}_f, \boldsymbol{\delta}_F, \delta_F^0$ involve in their integrands the factors $(\mathbf{x} - \mathbf{r})$, $\mathbf{B}_{\text{ex}}(t, \mathbf{x}) - \mathbf{B}_{\text{ex}}(t, \mathbf{r})$, $\mathbf{E}_{\text{ex}}(t, \mathbf{x}) - \mathbf{E}_{\text{ex}}(t, \mathbf{r})$ which vanish at the energy center $\mathbf{r}$, and they are small in a small neighborhood of $\mathbf{r}$. In addition to that, the integrands involve the factors $\mathbf{J}$,ρ which cannot be large outside a small neighborhood of the energy center if the solution ψ is localized.

Under the assumption that the charge is localized we neglect the particle discrepancy terms $\boldsymbol{\delta}_f, \boldsymbol{\delta}_F, \delta_F^0$ obtaining the following limit system for $\mathbf{r}$ and E:

$$\partial_t \mathsf{E} = \partial_t \mathbf{r} \cdot \mathbf{f}_{\text{Lor}}(t, \mathbf{r}), \tag{52}$$

$$\partial_t \left(\frac{\mathsf{E}}{\mathrm{c}^2} \partial_t \mathbf{r} \right) = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}). \tag{53}$$

Equation (53) evidently has the form of the relativistic version of Newton's law of motion with the Lorentz force, namely

$$\partial_t (M \partial_t \mathbf{r}) = \mathbf{f}_{\text{Lor}}(t, \mathbf{r}), \tag{54}$$

provided the mass M is given by Einstein's formula

$$M = \frac{\mathsf{E}}{\mathrm{c}^2}. \tag{55}$$

Equation (52) has the form of the time-component for the relativistic point dynamics, see [PauRT, Section 29, 37], [Barut, Section II.1]. Let us derive now a relation between the inertial mass M of moving charge and the rest mass. Using (52) we readily obtain

$$M \partial_t \mathbf{r} \cdot \partial_t (M \partial_t \mathbf{r}) = M \partial_t \mathbf{r} \cdot q \mathbf{E}_{\text{ex}}(t, \mathbf{r}) = M \mathrm{c}^2 \partial_t M, \tag{56}$$

which implies the relation

$$M^2 - \frac{1}{\mathrm{c}^2} M^2 (\partial_t \mathbf{r})^2 = M_0^2, \tag{57}$$

where M_0^2 is the constant of integration. Consequently, we recover the well-known formula

$$M = \gamma M_0, \quad \gamma = \left(1 - \left(\partial_t \mathbf{r}\right)^2 / \mathrm{c}^2\right)^{-1/2}. \tag{58}$$

The relations (54) and (58) readily imply the accelerated motion equation (1). As to the equations (52) and (53), they describe asymptotic behavior of the energy and the ergocenter of the charge when its wave function ψ remains localized in the course of motion.

Remark 2 *We would like to stress again that the rest mass M_0 in our treatment is not a prescribed quantity, but it is derived in (57) as an integral of motion (or, more precisely, an approximate integral of the field equation which becomes precise in an asymptotic limit). As any integral of motion, it can take different values for different "trajectories" of the field. This is demonstrated by the different values of the rest mass M_0 for different rest states constructed in the following Section 2.3. The integral of motion M_0 can be related to the mass m_0 of one of resting charges considered in Section 2.4 by the identity*

$$M_0 = m_0 \tag{59}$$

if the velocity vanishes on a time interval or asymptotically as $t \to -\infty$ or $t \to \infty$. If the velocity $\partial_t \mathbf{r}$ vanishes just at a time instant t_0 it is, of course, possible to express the value of M_0 in terms of $\mathsf{E} = \mathsf{E}(\psi)$ by formulas (55), (57) but the corresponding $\psi = \psi(t_0)$ may have no relation to the rest solutions of the field equation with a time independent profile $|\psi|^2$. It is also possible that $\partial_t \mathbf{r}$ never equals zero, and in fact this is a general case since all three components of velocity may vanish simultaneously only in very special situations. Hence, there is a possibility of localized regimes where the value of the "rest mass" M_0 may differ from the rest mass of a free charge. In such regimes the value of the rest mass cannot be derived based on the analysis of the uniform motion as in Section 2.4. This wide variety of possibilities makes even more remarkable the fact that the inertial mass is well-defined and that the Einstein formula (4) holds even in such general regimes where the standard analysis based on the Lorentz invariance of the uniform motion as in Section 2.4 does not apply. In a general case where the localization is not assumed, the functional

$$\hat{M}_0^2 = \frac{\mathsf{E}^2}{\mathrm{c}^4}\left(1 - \frac{1}{\mathrm{c}^2}\left(\partial_t \mathbf{r}\right)^2\right) \tag{60}$$

extends formula (57) to general fields and produces a "generalized rest mass" M_0 defined in terms of the energy and the ergocenter as follows:

$$M_0^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \hat{M}_0^2(t)\, dt. \tag{61}$$

The above formula obviously defines the value of the rest mass for a more general class of field trajectories, and according to (55) and (57) produces the value of the integral of motion in the case of asymptotic localization.

Remark 3 *An accelerating particle in an external field is not a closed system, and there are principal differences between closed and non-closed systems. In particular, the total momentum and the energy of a closed system are preserved. For closed systems the particle equation and the momentum kinematic representation can be derived from the field theory with the use*

of the angular momentum conservation, [Mol, Sec. 7.1, 72], [Sexl, Sec. 10.2], [LanVPM, Sec. 23]. For non-closed systems the center of energy (also known as center of mass or centroid) and the total energy-momentum are frame dependent and hence are not 4-vectors, [Mol, Sec. 7.1, 7.2], [LanVPM, Sec. 24]. The rules of the transformation for the energy-momentum are due to Einstein and Laue, [PauRT, Sec. 43].

2.3 Rest states, their energies and frequencies

We suppose for a resting charge $|\psi(t,x)|$ to be time independent. Such resting states of the charge exist in the absence of external fields, $\varphi_{\rm ex}=0$, $\mathbf{A}_{\rm ex}=0$. We are particularly interested in rest states ψ that vary harmonically in time and consider solutions to the KG equation (5) in the form of a standing wave

$$\psi = e^{-i\omega t}\breve{\psi}(x), \tag{62}$$

where $\breve{\psi}(x)$ is central-symmetric. The substitution of (62) in the KG equation (5) yields the following *nonlinear eigenvalue problem*

$$\nabla^2\breve{\psi} = G_a'\left(\left|\breve{\psi}\right|^2\right)\breve{\psi} + \left(\frac{\omega_0^2}{\mathrm{c}^2} - \frac{\omega^2}{\mathrm{c}^2}\right)\breve{\psi} = 0. \tag{63}$$

Recall now that the solution $\breve{\psi}$ must also satisfy the charge normalization condition (24) which takes the form

$$\int\left|\breve{\psi}\right|^2 \mathrm{d}x = \frac{\omega_0}{\omega}, \qquad \omega_0 = \frac{m\mathrm{c}^2}{\chi}. \tag{64}$$

The energy defined by (36), (38) yields for a standing wave (62) the following expression

$$\mathsf{E} = \frac{\chi^2}{2m}\int\left[\frac{1}{\mathrm{c}^2}\omega^2\breve{\psi}\breve{\psi}^* + \kappa_0^2\breve{\psi}\breve{\psi}^* + \nabla\breve{\psi}\nabla\breve{\psi}^* + G_a\left(\breve{\psi}\breve{\psi}^*\right)\right]\mathrm{d}x. \tag{65}$$

The problem (63), (64) has a sequence of solutions with the corresponding sequence of frequencies ω. Their energies $\mathsf{E}_{0\omega}$ are related to the frequency ω by the formula

$$\mathsf{E}_{0\omega} = \chi\omega\left(1+\Theta(\omega)\right), \qquad \Theta(\omega) = \frac{a_{\rm C}^2}{2a^2}\frac{\omega_0^2}{\omega^2}, \qquad a_{\rm C} = \frac{\chi}{m\mathrm{c}}, \tag{66}$$

where $a_{\rm C} = \frac{\lambda_{\rm C}}{2\pi}$ is the *reduced Compton wavelength* of a particle with a mass m and from now on we assume that $\chi = \hbar$.

The above expression for the energy involves the size parameter a. It coincides with the size of a *free* electron in the absence of EM fields. If our electron is placed in a strong EM field, its wave function ψ significantly changes. In particular, in the hydrogen atom the Coulomb field of a proton causes ψ to become a perturbation of an eigenstate of the linear hydrogen problem, [BF6]. Therefore, its actual size in the hydrogen atom is of order of the Bohr radius $a_{\rm B}$. It is physically reasonable to assume that the Coulomb field of the proton causes the distributed charge of the free electron to shrink, therefore a should be larger than $a_{\rm B}$. An analysis shows that for quantitative agreement with the classical hydrogen spectrum the value of a for the electron should be at least $100a_{\rm B}$, [BF6]; therefore, in (66) $a_{\rm C}^2/a^2 = \alpha^2 a_{\rm B}^2/a^2 \lessapprox 10^{-8}$. This relatively large size of a contrasts sharply with the concept of a point charge, but agrees well with physical properties of electron both at atomic and

macroscopic scales. Namely, for a larger value of a the nonlinearity becomes smaller, and the charge distribution at atomic scales becomes closer to the De Broglie wave, [BF4]. Note also that the electrostatic potential generated by the free charge with the size a is very close to the exact Coulomb potential at a macroscopic distance R from the center of the charge, namely the difference is of order e^{-R^2/a^2} and is extremely small if R is greater than a.

The Gaussian wave function

$$\psi\left(t,\mathbf{x}\right)=\mathrm{e}^{-\mathrm{i}\omega_0 t}a^{-3/2}\pi^{-3/4}\mathrm{e}^{-|x|^2/2a^2} \tag{67}$$

with $\omega=\omega_0$ is the *ground state* to the problem (it is referred to as gausson in [Bia]). This state has the minimal energy among all functions satisfying (64), hence it is stable. In (66) $a\gg a_{\mathrm{C}}$, and $\omega\geq\omega_0$ since ω_0 corresponds to the ground state. Therefore, *the relation (66) differs only slightly from the Planck-Einstein energy-frequency relation* $\mathsf{E}=\hbar\omega$. Note that in the non-relativistic version of our theory in ([BF6]) the relation $\Delta\mathsf{E}=\hbar\Delta\omega$ is an exact identity for hydrogenic atoms.

By a change of variables the original nonlinear eigenvalue problem (63) can be reduced to the following nonlinear eigenvalue problem with a logarithmic nonlinearity with only one eigenvalue parameter ξ and a parameter-independent constraint:

$$\nabla^2\breve{\psi}_1=G_1'\left(\left|\breve{\psi}\right|^2\right)\breve{\psi}_1-\xi\breve{\psi}_1,\ \int\left|\breve{\psi}_1\right|^2\,\mathrm{d}x=1. \tag{68}$$

The parameter ξ is related to the parameters in (63) by the formula

$$\xi=\frac{a^2}{a_{\mathrm{C}}^2}\left(\frac{\omega^2}{\omega_0^2}-1\right)-\frac{1}{2}\ln\frac{\omega^2}{\omega_0^2}. \tag{69}$$

The eigenvalue problem (68) has infinitely many solutions (ξ_n,ψ_{1n}), $n=0,1,2...$, representing localized charge distributions. The energy of ψ_n, $n>0$, is higher than the energy of the Gaussian ground state which corresponds to $\xi=\xi_0=0$ and has the lowest possible energy. These solutions coincide with critical points of the energy functional under the constraint, for mathematical details see [Caz], [BerLioI], [BerLioII]. The next two values of ξ for the radial rest states are approximately 2.17 and 3.41 according to [Bia1]. Putting in (69) the values $\xi=\xi_n$ we can find the corresponding values of $\frac{\omega_0}{\omega}$, yielding for $a^2\gg a_{\mathrm{C}}^2$ the following approximate formula

$$\frac{\omega_0}{\omega}\simeq 1-\xi\frac{a_{\mathrm{C}}^2}{2a^2}. \tag{70}$$

The difference of the energy of the higher states and the ground state energy is small, it is of order $\hbar\omega_0\xi\frac{a_{\mathrm{C}}^2}{2a^2}=\xi\frac{a_{\mathrm{B}}^2}{a^2}\hbar cR_\infty$ where $\hbar cR_\infty=mc^2\frac{\alpha^2}{2}$ is the Rydberg energy. Since we assume that $a_{\mathrm{B}}/a\lessapprox 10^{-4}$ the difference is comparable with the magnitude of the fine structure in the hydrogen atom which is of order $mc^2\alpha^4$. This comparison shows that the cohesive forces generated by the nonlinearity are relatively small.

Note that using the Lorentz invariance of the system one can easily obtain a solution which represents the charge-field moving with a constant velocity $\mathbf{v}$ simply by applying to the rest solution $(\psi,\varphi,\mathbf{0})$ the Lorentz transformation (see [BF4], [BF5] and the following Section 2.4).

Remark 4 *The rest states of higher energies corresponding $\xi=\xi_n$ with $n>0$ are unstable. Since the charges are coupled via electromagnetic interactions, there is a possibility of the*

energy transfer from charges with higher energies to the EM field, making such states very improbable in normal circumstances. It is conceivable though that in a system consisting of very many strongly interacting charges, such as, for instance, an astronomical object, a significant quantity of such states may be present contributing to the total energy and mass of such a system.

2.4 Uniform motion of a charge

Consider now a free motion of a charge governed by the KG equation (5) where the external fields vanish, that is $\varphi_{\text{ex}} = 0$, $\mathbf{A}_{\text{ex}} = 0$. Since the KG equation is relativistic invariant, the solution can be obtained from a rest solution defined by (62), (63) by applying Lorentz transformation as in [BF4], [BF6]. Hence, the solution to the KG equation (5) which represents a free particle that moves with velocity $\mathbf{v}$ is given by the formula

$$\psi(t,\mathbf{x}) = \psi_{\text{free}}(t,\mathbf{x}) = \mathrm{e}^{-\mathrm{i}(\gamma\omega t - \mathbf{k}\cdot\mathbf{x})}\breve{\psi}(\mathbf{x}'), \tag{71}$$

with $\breve{\psi}(\mathbf{x}')$ satisfying the equation (63), and

$$\breve{\psi}(\mathbf{x}') = \breve{\psi}_a(\mathbf{x}') = a^{-3/2}\breve{\psi}_1(\mathbf{x}'/a), \tag{72}$$

$$\mathbf{x}' = \mathbf{x} + \frac{(\gamma-1)}{v^2}(\mathbf{v}\cdot\mathbf{x})\mathbf{v} - \gamma\mathbf{v}t, \quad \mathbf{k} = \gamma\omega\frac{\mathbf{v}}{\mathrm{c}^2}, \tag{73}$$

where γ is the Lorentz factor

$$\gamma = \left(1-\boldsymbol{\beta}^2\right)^{-1/2},\ \boldsymbol{\beta} = \frac{1}{\mathrm{c}}\mathbf{v}. \tag{74}$$

All characteristics of a free charge can be explicitly written. Namely, the charge density ρ defined by the relation (22) and the total charge E equal respectively

$$\rho = \gamma q\left|\breve{\psi}(\mathbf{x}')\right|^2, \quad \bar{\rho} = \int \rho(\mathbf{x})\ \mathrm{d}\mathbf{x} = q, \tag{75}$$

$$\mathsf{E} = \gamma m\mathrm{c}^2(1+\Theta(\omega)), \tag{76}$$

where $\Theta(\omega)$ is given by (66). The current density$\mathbf{J}$, the total momentum $\mathbf{P}$ and the total current $\bar{\mathbf{J}}$ for the free charge equal respectively

$$\mathbf{J} = \frac{q}{m}\hbar\,\mathrm{Im}\frac{\nabla\psi}{\psi}|\psi|^2 = \gamma\hbar\frac{q}{m}\omega\frac{1}{\mathrm{c}^2}\mathbf{v}\left|\breve{\psi}\right|^2(\mathbf{x}'), \tag{77}$$

$$\mathbf{P} = \gamma m\mathbf{v}(1+\Theta(\omega)), \quad \bar{\mathbf{J}} = \int \mathbf{J}(\mathbf{x})\ \mathrm{d}\mathbf{x} = q\mathbf{v}. \tag{78}$$

The 4-vector $(\mathsf{E},\mathbf{P})$ is a relativistic energy-momentum 4-vector with the Lorentz invariant $\mathsf{E}^2 - \mathrm{c}^2\mathbf{P}^2 = (1+\Theta)^2 m^2\mathrm{c}^4$. Hence, based on the commonly used argument, [Mol, Sec. 3.3], [PauRT, Sec. 37], it is natural to define the *rest mass* of the charge in terms of the mass parameter m by the formula

$$m_0 = m(1+\Theta(\omega)). \tag{79}$$

A direct comparison shows that the above definition based on the Lorentz invariance of uniformly moving free charge is fully consistent with the definition of the inertial mass which was derived from the analysis of accelerated motion of localized charges in external EM field in the previous subsection, see Remark 2 for a more detailed discussion.

2.5 The spectroscopic and inertial masses

Our theory produces a description of the hydrogen atom, [BF6], [BF7], yielding in the non-relativistic case an asymptotic formula $E_n = -\hbar c R_\infty / n^2$ for the hydrogen energy levels with the factor R_∞ given by the formula

$$R_\infty = \frac{q^4 m}{2\hbar^3 \mathrm{c}}. \tag{80}$$

In the relativistic case a more complex formula can be derived which involves an equivalent of the Sommerfeld fine structure with the same factor $\hbar c R_\infty$. The constant R_∞ in (80) coincides with the expression for the Rydberg constant if $m = m_\mathrm{e}$ is the electron mass and q equals the electron charge. It seems, therefore, natural to refer to the mass parameter m as the *spectroscopic mass*. We make a distinction between the spectroscopic mass and the inertial mass since in our theory the mass parameter m of a charge is somewhat smaller than the inertial mass m_0 defined by the formula (79) with $\omega = \omega_0$, namely

$$m_0 = m\left(1 + \frac{1}{2}\frac{a_\mathrm{C}^2}{a^2}\right),\ a_\mathrm{C} = \frac{\hbar}{m\mathrm{c}}. \tag{81}$$

The difference $m_0 - m$ depends evidently on the size parameter a. The question stands now: is there any experimental evidence which shows that the inertial mass m_0 and the spectroscopic mass m may be different as in our theory?

The quantum mechanics and the quantum electrodynamics allow to interpret the spectroscopic data and extract from it the mass of electron known as the *recommended value of the electron mass* $m = m_\mathrm{e} = A_\mathrm{r}(e)$. The recommended value of the electron mass $A_\mathrm{r}(e)$ when expressed in units u, can be found in [Mohr6, Table XLIX, p. 710]:

$$A_\mathrm{r}(e) = 5.4857990943\,(23) \times 10^{-4}\ \left[4.2 \times 10^{-10}\right]\ \text{(recommended value)}, \tag{82}$$

where, we remind the common convention, 5.4857990943 represents the mean value of the experimental data, 0.0000000023 represents its standard uncertainty (deviation) and 4.2×10^{-10} represents its relative uncertainty. The value $A_\mathrm{r}(e)$ is found based on very extensive spectroscopic data.

But there is also another class of measurements, namely the Penning trap measurements, which can be considered as the most direct measurement of the electron mass as the inertial one, and it gives the following mass value, [Far]:

$$A_\mathrm{r}(e) = 5.485799111\,(12) \times 10^{-4}\ \left[2.1 \times 10^{-9}\right]\ \text{(Penning trap)}. \tag{83}$$

Observe now that the value of the electron inertial mass coming from the Penning trap measurement (83) is larger than the spectroscopic mass (82), and the difference is statistically significant. Indeed, using standard statistical analysis we obtain for the difference $\delta A_\mathrm{r}(e) = m_0 - m$ the following

$$m_0 - m = \delta A_\mathrm{r}(e) = 0.17(12) \times 10^{-11},\ \left[3.1 \times 10^{-9}\right]. \tag{84}$$

If we take the *recommended value* $A_\mathrm{r}(e)$ from (82) as our "golden standard" then the mean value of the Penning trap measurements from (83) is more than 7 standard deviations above the mean recommended value $A_\mathrm{r}(e) = 5.4857990943$, and the same is true for approximately

one half of all Penning trap measurements which are greater than the mean value. *This simple statistical calculation makes our point, namely that the inertial mass m_0 as represented by Penning trap measurements (83) is larger than the recommend value m from (82).*

We have not succeeded in finding in the existing literature an explanation of the above mass difference, but in our theory such a difference including its positive sign can be easily explained by the finite value of the electron size parameter a. The resulting relation between the two masses is given by the formula (81) which can be recast as

$$\frac{a_{\rm C}^2}{a^2} = 2\frac{m_0 - m}{m}. \tag{85}$$

From relations (84) and (82) we obtain approximate inequalities

$$0.87 \times 10^{-9} \preceq \frac{m_0 - m}{m} \preceq 0.53 \times 10^{-8}. \tag{86}$$

If we assign the effect to the non-zero value of a, using the formula (85) we obtain

$$0.97 \times 10^{4} \preceq \frac{a}{a_{\rm C}} \preceq 2.4 \times 10^{4}. \tag{87}$$

These inequalities are consistent with our prior assessments for a to be at least of order $10^2\alpha^{-1}a_{\rm C}$. The mentioned assessments are made in [BF5], [BF6] based on the analysis of the frequency spectrum of the hydrogen atom, see also a discussion after relation (66). Note that in the derivation of (87) we used an asymptotic formula for the spectrum from our theory of the hydrogen atom. Consequently, the terms which were neglected may result in a much larger interval for $a/a_{\rm C}$, and, hence, the relations (87) are more for illustration purposes rather than for an actual estimate of the ratio $a/a_{\rm C}$.

2.6 Charge localization assumptions

The ergocenter obeys the relativistic version of Newton's equation if the particle discrepancy terms $\boldsymbol{\delta}_f, \boldsymbol{\delta}_F, \delta_F^0$ in the ergocenter equations (51), (50) can be neglected. These terms would vanish exactly if the charge and corresponding currents are localized exactly at the center $\mathbf{r}$, or if $\mathbf{r}$ is a center of symmetry and the external EM potentials are constant as in the case of a uniformly moving charge. In the general case we may only expect that these terms vanish asymptotically in a certain limit, namely,

$$\boldsymbol{\delta}_f \to 0, \quad \boldsymbol{\delta}_F \to 0, \quad \delta_F^0 \to 0. \tag{88}$$

There are two kinds of quantities which enter the discrepancy terms. One kind involves differences $\mathbf{E}_{\rm ex}(t,\mathbf{x}) - \mathbf{E}_{\rm ex}(t,\mathbf{r})$ and $\mathbf{B}_{\rm ex}(t,\mathbf{x}) - \mathbf{B}_{\rm ex}(t,\mathbf{r})$; they vanish if the fields are constant and are small if the fields are almost constant. The magnitude of inhomogeneity of the EM fields can be described by the typical length R_f at which they vary significantly. Hence, to ensure that the fields are almost constant near the charge, we assume that R_f is much larger than the charge size a and impose on the external field the following *asymptotic local homogeneity* condition:

$$a/R_f \to 0. \tag{89}$$

Another kind of quantities which enter the discrepancy terms involve factors $(\mathbf{x}-\mathbf{r})$ as in $\boldsymbol{\delta}_F$ in (49):

$$\int \frac{1}{\mathrm{c}}\left(\partial_t\left(\left(\mathbf{x}-\mathbf{r}\right)\rho\right)+\sum_l \partial_l\cdot\left(\left(\mathbf{x}-\mathbf{r}\right)\mathbf{J}^l\right)\right)\times\mathbf{B}_{\mathrm{ex}}\mathrm{d}\mathbf{x}$$
$$=\frac{1}{\mathrm{c}}\partial_t\int\rho\left(\mathbf{x}-\mathbf{r}\right)\times\mathbf{B}_{\mathrm{ex}}\mathrm{d}\mathbf{x}-\frac{1}{\mathrm{c}}\int\rho(\mathbf{x}-\mathbf{r})\times\partial_t\mathbf{B}_{\mathrm{ex}}d\mathbf{x}+\int\frac{1}{\mathrm{c}}\int\sum_l\mathbf{J}^l\left(\mathbf{x}-\mathbf{r}\right)\times\partial_l\mathbf{B}_{\mathrm{ex}}\mathrm{d}\mathbf{x}.$$

These quantities vanish for spatially constant $\mathbf{B}_{\mathrm{ex}},\mathbf{E}_{\mathrm{ex}}$ if ρ and every component of $\mathbf{J}$ is center-symmetric with respect to $\mathbf{r}$. Similar quantities are present in δ_F^0 and $\boldsymbol{\delta}_f$ defined by (43), (45). For example

$$\delta_f^i=\int\left(x^i-r^i\right)\left(\rho\mathbf{E}_{\mathrm{ex}}+\frac{1}{\mathrm{c}}\mathbf{J}\times\mathbf{B}_{\mathrm{ex}}\right)^i\mathrm{d}\mathbf{x} \tag{90}$$

vanish under the same central symmetry assumption. Hence, to satisfy the charge localization condition (88), it is sufficient to impose two separate requirements: (i) the asymptotic condition (89) and (ii) asymptotic central symmetry of ρ and components of $\mathbf{J}$.

To clarify the meaning of the above localization conditions, let us look at the simplest case where ρ and $\mathbf{J}$ are derived for the uniform motion of a free particle described by the solution $\psi\left(t,\mathbf{x}\right)=\psi_{\mathrm{free}}\left(t,\mathbf{x}\right)$ considered in Section 2.4. The solution has the following properties:

(i) the energy density u is center-symmetric with respect to $\mathbf{r}\left(t\right)=\mathbf{v}t$, hence the ergocenter coincides with $\mathbf{r}\left(t\right)$;

(ii) the charge density ρ is given by (75), and according to (72) it converges to $q\delta\left(\mathbf{x}-\mathbf{r}\right)$ as $a\to 0$ where $\delta\left(\mathbf{x}\right)$ is the Dirac delta-function;

(iii) the current $\mathbf{J}$ is given by (77), its components are center-symmetric and converge to the corresponding components of $q\mathbf{v}\delta\left(\mathbf{x}-\mathbf{r}\right)$.

Hence, the localization assumptions (88) are fulfilled for ρ and $\mathbf{P}$ derived for $\psi_{\mathrm{free}}\left(t,\mathbf{x}\right)$ for general fields $\mathbf{E}_{\mathrm{ex}},\mathbf{B}_{\mathrm{ex}}$ which are regular near $\mathbf{r}$ when $a/R_f\to 0$. If the motion is almost uniform, namely if the external fields are not too strong, and the solution $\psi\left(t,\mathbf{x}\right)$ of (5) is close to $\psi_{\mathrm{free}}\left(t,\mathbf{x}\right)$, we may expect that such a solution also satisfies the localization conditions, we present an example in Section 3.

Now let us briefly discuss condition (89). Suppose that the corresponding forces, according to (53), are of the order $\hat{f}_{\mathrm{Lor}}$ where $\hat{f}_{\mathrm{Lor}}$ is a typical magnitude of the Lorentz force $\mathbf{f}_{\mathrm{Lor}}\left(t,\mathbf{r}\right)$. The spatial scale R_f at which the forces associated with the electromagnetic fields $\mathbf{E}_{\mathrm{ex}},\mathbf{B}_{\mathrm{ex}}$ vary by the same order of magnitude as $\hat{f}_{\mathrm{Lor}}$ can be defined as follows:

$$\frac{1}{R_f}=\frac{q}{\hat{f}_{\mathrm{Lor}}}\max_{|x-r|\leq\theta a}\left(\frac{1}{\mathrm{c}}\left|\nabla\mathbf{B}_{\mathrm{ex}}\right|+\left|\nabla\mathbf{E}_{\mathrm{ex}}\right|\right) \tag{91}$$

with $\theta\gg 1$. In view of this definition, condition (89) ensures that the variability of EM fields causes a vanishing perturbation to the Lorentz force in (53).

The estimates of smallness of the discrepancy terms can be made in certain asymptotic regimes, but they are laborious, and we would like to make some guiding comments. The KG equation (93) evidently involves five parameters, namely, $\mathrm{c},\hbar,m,\,q,a$, and the external fields also can depend on parameters. The limits (88) can be considered simultaneously with certain combinations of the parameters tending to their limits, and the limits in (88) have

to be taken together with these parameter limits. Note that all the quantities which enter equations (50) (51) are the integrals of certain densities over the entire space. The integrals over the entire space obviously are the limits of the integrals over the domains $|\mathbf{x}-\mathbf{r}|<\theta a$ with $\theta\to\infty$, and the value of θ in the asymptotic regimes can be related with the values of the other parameters mentioned. An example of such asymptotic situation with nontrivial relations between the parameters is given in the following section.

3 Relativistic accelerated motion of a wave-corpuscle

Up to now, we were considering relativistic features of an accelerating charge assuming its localization. The localization assumptions, though natural, are rather technical when it comes to rigorous treatment. With that in mind, we study in this section a particular case where the dynamical problem is non-trivial and produces a wide variety of accelerated relativistic motions of the charge which are simple enough for a detailed analysis and the verification of the localization assumptions. The analysis is still rather involved, and that comes at no surprise since for general external EM fields the KG equation (5) has no closed form solutions. We succeeded though in finding a large family of non-trivial regimes for which we obtain almost explicit representations of solutions allowing for a detailed study of the relativistic features of the charge accelerating in an external EM field.

3.1 Rectilinear charge motion

In the previous section we studied dynamics of localized accelerating waves in general EM fields. The relativistic point dynamics was derived under the assumption that the localization conditions (88) hold. Here we present an example of an accelerating charge when its localization can be maintained in the strongest possible form. Namely, we consider a regime when the accelerating charge *exactly preserves its Gaussian shape* up to the Lorentz contraction; we call such a solution of the KG equation a wave–corpuscle. The charge wave function is similar to $\psi_{\text{free}}(t,\mathbf{x})$ in (71) and consequently it is localized about $\mathbf{r}(t)$, but its velocity $\mathbf{v}=\partial_t\mathbf{r}$ is not constant and the charge has a non-zero acceleration. *The shape of the wave in an accelerated regime is exactly the same as for a free particle, and it is only the phase factor that is affected by the acceleration caused by the external force.* Such an accelerated motion with a fixed shape is possible only for a properly chosen potential φ_{ex}. We consider here the simplest but still non-trivial case where the charge moves and accelerates in the direction of the axis x_3 with the potential φ_{ex} being a function of only the variable x_3 and the time t, and there is no external magnetic field, that is $\mathbf{A}_{\text{ex}}=0$.

When the external potential φ_{ex} depends only on t,x_3, the equation (5) in the three dimensional space with a logarithmic nonlinearity (12) can be exactly reduced to a problem in one dimensional space by the following substitution

$$\psi=\pi^{-1/2}a^{-1}\exp\left(-\frac{1}{2a^2}\left(x_1^2+x_2^2\right)\right)\psi_{1D}(t,x_3)\,, \tag{92}$$

We obtain a reduced 1D KG equation for $\psi=\psi_{1D}(t,x_3)$ with *one spatial variable*:

$$-\frac{1}{\mathrm{c}^2}\tilde{\partial}_t\tilde{\partial}_t\psi+\partial_3^2\psi-G'_{a1}(\psi^*\psi)\,\psi-\kappa_0^2\psi=0. \tag{93}$$

where

$$\kappa_0 = \frac{mc}{\hbar}, \quad \tilde{\partial}_t = \partial_t + \frac{iq}{\hbar}\varphi_{ex}, \quad \varphi_{ex} = \varphi_{ex}(t, x_3),$$

and the $1D$ logarithmic nonlinearity has the form

$$G'_{a1}\left(|\psi|^2\right) = -a^{-2}\left[\ln\left(\pi^{1/2}|\psi|^2\right) + 1\right] - a^{-2}\ln a. \tag{94}$$

This equation has a Gaussian as a rest solution,

$$\breve{\psi}(x_3) = \pi^{-1/4}a^{-1/2}e^{-x_3^2/2a^2}. \tag{95}$$

From now on we write x instead of x_3 for the notational simplicity.

We consider the center location $r(t)$ to be a given function. We also assume that the motion is translational. Namely, the solution $\psi(t,x)$ to (93) has the following Gaussian form:

$$\psi = a^{-1/2}e^{-iS_0(t,y)}\mathring{\psi}(a^{-1}\gamma_0 y), \quad y = x - r(t), \tag{96}$$
$$\mathring{\psi}(t,z) = \pi^{-1/4}\gamma^{-1/2}\gamma_0^{1/2}e^{-z^2/2},$$

where $\gamma = \gamma(t)$ is the Lorentz factor, $\gamma_0 = \gamma(0)$, the phase $S_0(t,y)$ has to be determined. We look for such a potential φ_{ex} that the wave ψ defined by (96) is an exact solution of the KG equation (93). If such a potential is found, the function $\psi(t,x)$ describes an accelerating charge with the strongest possible localization, namely, a charge with a fixed shape $|\psi|$ and arbitrarily small size a.

3.2 Mild acceleration regime

Below we provide an example of a regime which allows simultaneously localization and acceleration, with a general charge trajectory $r(t)$ subjected to certain regularity conditions. The KG equation (93) involves five parameters, namely the speed of light c, Planck constant $\hbar$, the mass parameter m, the charge value q and the charge size a. A combination of the parameters which is important in our analysis is the reduced Compton wavelength $a_C = \frac{\hbar}{mc}$. Now we describe relations between the parameters which allow for localized accelerating charges.

We begin with introducing a class of admissible trajectories $r(t)$. We assume that the charge does not undergo violent accelerations and suppose a regular dependence of the normalized velocity β on the dimensionless time $\tau = ct/a$, namely

$$\max_\tau\left(|\partial_\tau\beta| + \left|\partial_\tau^2\beta\right| + \left|\partial_\tau^3\beta\right|\right) \le \epsilon, \quad \text{where } \tau = \frac{c}{a}t. \tag{97}$$

We also assume that the velocity $v = \partial_t r$, as well as its variation, is smaller than the speed of light c, namely

$$\frac{1}{c}\max_t|v(t)| \le \epsilon_1, \qquad \frac{1}{c}\max_t|v(t) - v(0)| \le \epsilon_1 \qquad \epsilon_1 < 1. \tag{98}$$

Obviously, the variation of β and of the Lorentz factor γ remains bounded for all τ, that is

$$|\gamma_0 - \gamma(\tau)| \le C\epsilon_1, \qquad |\beta_0 - \beta(\tau)| \le \epsilon_1, \tag{99}$$

where $\gamma_0 = \gamma(0)$, $\beta_0 = \beta(0)$. Observe that the above assumptions are consistent with significant changes of velocity v in asymptotic regimes where $\mathrm{c} \to \infty$, $v \to \infty$, $v/\mathrm{c} \to$ const. Note that condition (99) does not follow from (97). It can can be seen from the following example of oscillatory motion clarifying the above conditions.

Let $v(t)$ be defined by

$$v(t) = \mathrm{c}\beta_0 + \bar{v}_1 \sin^2\left(\frac{\mathrm{c}}{a}\eta t\right) = \mathrm{c}\beta_0 + \bar{v}_1 \sin^2(\eta\tau) \text{ for } t \geq 0, \tag{100}$$
$$v(t) = \mathrm{c}\beta_0 \text{ for } t \leq 0,$$

where $\bar{v}_1$ is the amplitude of the variable part of velocity, η is a dimensionless frequency, $\beta_0 < 1$ is the initial normalized velocity. We readily obtain the following estimate of the normalized velocity variation:

$$\frac{1}{\mathrm{c}}|v(t) - v(0)| \leq \frac{1}{\mathrm{c}}\bar{v}_1,$$

and to satisfy (98) and (97) we set

$$\frac{1}{\mathrm{c}}\bar{v}_1 = \epsilon_1, \qquad \frac{1}{\mathrm{c}}\bar{v}_1\left(\eta + \eta^2 + \eta^3\right) = \epsilon. \tag{101}$$

Since the parameter η can depend on ϵ_1, the boundedness of ϵ_1 and ϵ do not follow one from another. The above conditions imply a bound on the acceleration,

$$|\partial_t v| = \eta\bar{v}_1 \mathrm{c} a^{-1} |\sin(2\eta \mathrm{c}t/a)| \leq \epsilon \mathrm{c}^2/a,$$

which involves a large factor c^2/a; therefore, the acceleration can be large even if the parameter ϵ is small.

Let us consider now the following important *dynamical problem*: determine the external electric field which generates a purely translational motion described by (96). The size parameter a sets up a natural microscopic length scale. We assume that the velocity $v(t)$ satisfies conditions (97) and (98), and the parameters of the problem satisfy the following two restrictions. First, we assume that the Compton wavelength defined in (66) is much smaller than the size parameter a, namely

$$\zeta = \frac{a_{\mathrm{C}}}{a} \ll 1. \tag{102}$$

The second condition relates ζ to the parameters ϵ and ϵ_1 from (97), (98):

$$\frac{\epsilon_1}{\epsilon}\zeta^2 \ll 1. \tag{103}$$

For a given trajectory $r(t)$ we find then a potential φ_{ex} such that the Gaussian wave function with the center $r(t)$ is an *exact* solution of (93) in the strip $\Xi(\theta) = \{|x - r(t)| \leq \theta a\}$ around the trajectory, and θ grows to infinity as $\zeta \to 0$. The solution ψ is similar to the free solution but it allows for an acceleration. We will show that the trajectory $r(t)$ relates to the potential according to the relativistic version of Newton's law

$$\partial_t(m\gamma v) + q\partial_x\varphi_{\mathrm{ac}}(t, r) = 0. \tag{104}$$

Here $\varphi_{\mathrm{ac}}(x)$ is the leading part of the potential φ_{ex} which causes the acceleration of the charge, we call it "accelerating" potential, it does not depend on the small parameter ζ. The

remaining part of the external potential $\varphi_{\rm ex}$ in the KG equation (93) is a small "balancing" potential

$$\varphi_2(t,x,\zeta)=\varphi_{\rm ex}(t,x;\ \zeta)-\varphi_{\rm ac}(t,x),$$

which allows the charge to exactly preserve its form as it accelerates. The balancing potential vanishes asymptotically, that is $\varphi_2(t,x;\zeta)\to 0$ as $\zeta\to 0$, and the forces it produces also become vanishingly small compared with the Lorentz force $q\partial_x\varphi_{\rm ac}(x)$ in the strip $\Xi(\theta)$. The potential $\varphi_{\rm ex}(t,x)$ converges to its accelerating part $\varphi_{\rm ac}$ in the strip $\Xi(\theta)$ with $\theta\to\infty$ as $\zeta\to 0$. In view of (104) it is natural to treat φ_2 as a perturbation which does not affect the acceleration, but is responsible for the exact preservation of the shape of charge distribution during its evolution. In the asymptotic limit $\zeta\to 0$ we have in (104)

$$\partial_x\varphi_{\rm ex}(r(t))\to\partial_x\varphi_{\rm ac}(t,r(t)). \tag{105}$$

The summary of what we intend to fulfill is as follows. For any given trajectory $r(t)$ we construct a potential $\varphi_{\rm ex}$ which makes the Gaussian wave with the center $r(t)$ to be an exact solution of the field equation in the widening strip $\Xi(\theta)$, $\theta\to\infty$. Since the shape of $|\psi|$ is preserved, such a function is localized around $r(t)$. We can show that in the considered regimes the typical spatial scale R of inhomogeneity of the constructed electric field tends to infinity, $R/a\to\infty$, and hence (89) is fulfilled.

The implementation of the outlined above approach is provided below and here is yet another look at it. We construct the external field which causes the accelerated motion of a charge distribution with a fixed Gaussian shape; the distribution exactly satisfies the relativistic covariant KG equation in a wide strip about the wave center trajectory. Consequently, it provides an example for the relativistic dynamics of a charge wave function with a fixed shape. The possibility of a uniform global motion without acceleration is well-known, see Section 2.4 and [BenFor]. The fact that relativistic acceleration imposes restrictions on the spatial extension of rigid bodies was noted in a different setting in [Eri82].

3.3 Equation in a moving frame

As the first step of the analysis we rewrite the KG equation (93) in a moving frame around r. We take $r(t)$ as the new origin and make the following change of variables

$$x_3=r(t)+y,\quad \psi(t,x)=\psi'(t,y),\quad v=\partial_t r. \tag{106}$$

The 1D KG equation (93) then takes the form

$$\begin{aligned}-\frac{1}{\mathrm{c}^2}\left(\partial_t+\frac{\mathrm{i}q\varphi_{\rm ex}}{\hbar}-v\partial_y\right)\left(\partial_t+\frac{\mathrm{i}q\varphi_{\rm ex}}{\hbar}-v\partial_y\right)\psi' \\ +\partial_3^2\psi'-G_a'(\psi'^*\psi')\psi'-\frac{1}{a_C^2}\psi'=0,\end{aligned} \tag{107}$$

where $v(t)$ is a given function of time. We write the external potential $\varphi_{\rm ex}$ which produces the motion in the form

$$\varphi_{\rm ex}(t,x)=\varphi_{\rm ac}(t,y)+\varphi_2(t,y,\zeta), \tag{108}$$

where the *accelerating potential is linear in* y

$$\varphi_{\rm ac}=\varphi_0(t)+\varphi_{\rm ac}'y,$$

and $\varphi_2(t,y)$ is a *balancing potential*, which, as we will show, is small. The 1D logarithmic nonlinearity $G'_a = G'_{a1}$ is defined by (94). We assume that the solution $\psi(t,x)$ has the Gaussian form, namely

$$\psi' = \mathrm{e}^{\mathrm{i}\frac{\omega_0}{\mathrm{c}^2}\gamma v y - \mathrm{i}s(t) - \mathrm{i}S(t,y)}\Psi', \quad \omega_0 = \frac{\mathrm{c}}{a_{\mathrm{C}}}, \tag{109}$$

where we define Ψ' by an explicit expression

$$\Psi' = a^{-1/2}\mathring{\psi}_1\left(a^{-1}\gamma_0 y\right), \tag{110}$$

where γ_0 is the Lorentz factor defined by (74) from (99) and

$$\mathring{\psi}_1(t,z) = \pi^{-1/4}\mathrm{e}^{\sigma - z^2/2}, \tag{111}$$

$$\sigma = \sigma(t) = \frac{1}{2}\ln\frac{\gamma_0}{\gamma}. \tag{112}$$

Using (99) and (97), we observe that σ is bounded, namely

$$\sigma = O(\epsilon_1), \quad \partial_\tau\sigma = O(\epsilon), \quad \partial_\tau^2\sigma = O(\epsilon). \tag{113}$$

Substitution of (109) into (107) produces

$$\begin{aligned} -\frac{1}{\mathrm{c}^2}\Bigg(&\partial_t + \mathrm{i}\partial_t(\gamma v)\frac{\omega_0}{\mathrm{c}^2}y - \mathrm{i}\partial_t s + \frac{\mathrm{i}q\varphi_0}{\hbar} - \mathrm{i}v^2\frac{\omega_0}{\mathrm{c}^2}\gamma \\ &-\mathrm{i}\partial_t S + \mathrm{i}v\partial_y S + \frac{\mathrm{i}q\varphi'_{\mathrm{ac}}}{\hbar}y + \frac{\mathrm{i}q\varphi_2}{\hbar} - v\partial_y\Bigg)^2\Psi' \\ +\left(\partial_y - \mathrm{i}\partial_y S + \mathrm{i}v\frac{\omega_0}{\mathrm{c}^2}\gamma\right)^2&\Psi' - G'_a\left(|\Psi'|^2\right)\Psi' - \kappa_0^2\Psi' = 0. \end{aligned} \tag{114}$$

We show below that φ_2 and S are of order $\zeta^2(\epsilon_1+\epsilon)$. To eliminate the leading (independent of the small parameter ζ) terms in the above equation, we require that the following two equations are fulfilled:

$$-\partial_t s + \frac{q\varphi_0}{\hbar} - v^2\frac{\omega_0}{\mathrm{c}^2}\gamma = -\gamma\omega_0, \tag{115}$$

and

$$m\partial_t(\gamma v) + q\varphi'_{\mathrm{ac}} = 0. \tag{116}$$

Obviously, the expressions in (114) eliminated via equations (115) and (116) do not depend on ζ. The equation (116) determines the essential part $\varphi'_{\mathrm{ac}}y$ of the accelerating potential; the constant part $\varphi_0(t)$ of the accelerating potential can be prescribed arbitrarily, since we always can choose the phase shift $s(t)$ so that (115) is fulfilled. Equation (116) evidently coincides with the *relativistic law of motion* (104).

The above conditions annihilate in the equation (114) all the terms which do not vanish as ζ tends to zero; we show this in the following two sections. As a first step, we simplify equation (107) using equations (115) and (116), and obtain the following equation:

$$\begin{aligned} -\frac{1}{\mathrm{c}^2}&\left(\partial_t - \mathrm{i}\gamma\frac{\mathrm{c}}{a_{\mathrm{C}}} - \mathrm{i}\partial_t S + \mathrm{i}v\partial_y S + \frac{\mathrm{i}q\varphi_2}{\hbar} - v\partial_y\right)^2\Psi' \\ +\left(\partial_y - \mathrm{i}\partial_y S + \mathrm{i}\frac{1}{a_{\mathrm{C}}}\frac{v}{\mathrm{c}}\gamma\right)^2&\Psi' - G'_a\left(|\Psi'|^2\right)\Psi' - \frac{1}{a_{\mathrm{C}}^2}\Psi' = 0. \end{aligned} \tag{117}$$

In the following sections we find small φ_2 and S which are of order $\zeta^2(\epsilon_1+\epsilon)$.

3.4 Equations for auxiliary phases

In this subsection we introduce two auxiliary phases, and then reduce the problem of determination of the potential to a simpler first-order partial differential equation for one unknown phase. Solving such an equation can be further reduced to integration along characteristics and allows a rather detailed mathematical analysis. Here we restrict ourselves to the simplest steps in the analysis, but the introduced construction can be used for a much more detailed analysis of the relativistic interaction of the EM field with a rigid charge.

It is convenient to introduce rescaled dimensionless variables z, τ:

$$\tau = \frac{\mathrm{c}}{a}t, \quad z = \frac{\zeta}{a_{\mathrm{C}}}y = \frac{1}{a}y, \quad \Psi = a^{1/2}\Psi', \tag{118}$$
$$\partial_y = \frac{1}{a}\partial_z = \frac{\zeta}{a_{\mathrm{C}}}\partial_z, \quad \partial_t = \frac{\mathrm{c}}{a}\partial_\tau = \zeta\frac{\mathrm{c}}{a_{\mathrm{C}}}\partial_\tau.$$

We introduce auxiliary phases Z and Φ,

$$Z = \zeta\partial_z S, \tag{119}$$
$$\Phi = -\zeta\partial_\tau S + \zeta\beta\partial_z S + \frac{qa_{\mathrm{C}}\varphi_2}{\mathrm{c}\hbar}; \tag{120}$$

they will be our new unknown variables. These auxiliary phases determine the balancing potential φ_2 which we intend to make vanishingly small. Obviously, if we find Z and Φ, we can find S by the integration in z and we set $S = 0$ at $z = 0$. After that φ_2 can be found from (120). Consequently, to find a small φ_2 we need to find small Z and Φ.

Equation (117) takes the following form:

$$-\left(\zeta\partial_\tau + \mathrm{i}\Phi - \mathrm{i}\gamma - \beta\zeta\partial_z\right)^2\Psi \tag{121}$$
$$+\left(\zeta\partial_z - \mathrm{i}Z + \mathrm{i}\beta\gamma\right)^2\Psi - \zeta^2 G_1'\left(\Psi^*\Psi\right)\Psi - \Psi = 0.$$

We look for a solution of (121) in the strip Ξ in time-space:

$$\Xi = \{(\tau, z) : -\infty \le t \le \infty, \quad |z| \le \theta\} \tag{122}$$

where θ is a large number. Note that $|\Psi| = \pi^{-1/4}\mathrm{e}^{-\gamma_0^2 z^2/2}$ is smaller than $\pi^{-1/4}\mathrm{e}^{-\theta^2/2}$ outside Ξ and is extremely small for large θ, and later we are going to make θ arbitrarily large. We expand (121) with respect to Φ, Z yielding

$$Q\Psi - \mathrm{i}\Phi\left(\zeta\partial_\tau - \mathrm{i}\gamma - \beta\zeta\partial_z\right)\Psi - \mathrm{i}\left(\zeta\partial_\tau - \mathrm{i}\gamma - \beta\zeta\partial_z\right)(\Phi\Psi) + \Phi^2\Psi \tag{123}$$
$$+\mathrm{i}Z\left(\zeta\partial_z + \mathrm{i}\beta\gamma\right)\Psi + \mathrm{i}\left(\zeta\partial_z + \mathrm{i}\beta\gamma\right)(Z\Psi) - Z^2\Psi = 0,$$

where we denote by $Q\Psi$ the term which does not involve Φ and Z explicitly:

$$Q = \frac{1}{\Psi}\left(-\left(\zeta\partial_\tau - \mathrm{i}\gamma - \beta\zeta\partial_z\right)^2\Psi + \left(\zeta\partial_z + \mathrm{i}\beta\gamma\right)^2\Psi\right) - \zeta^2 G_1'\left(\Psi^*\Psi\right) - 1. \tag{124}$$

Since (112) holds, the imaginary part of Q is zero,

$$\operatorname{Im} Q = 2\zeta\gamma\partial_\tau\sigma + \zeta\partial_\tau\gamma = 0. \tag{125}$$

Hence Q is real and we have

$$Q = \operatorname{Re} Q = \left(-\frac{1}{\Psi}\partial_\tau^2 \Psi + \partial_\tau \beta \frac{1}{\Psi}\partial_z \Psi + \frac{2}{\Psi}\beta \partial_\tau \partial_z \Psi + \frac{1}{\gamma^2}\frac{1}{\Psi}\partial_z^2 \Psi - G_1'\left(\Psi^2\right)\right)\zeta^2. \tag{126}$$

By (94) with $\Psi^2 = \mathrm{e}^{2\sigma}\Psi_1^2$, we obtain that

$$G_1'\left(\mathrm{e}^{2\sigma}\Psi_1^2\right) = -2\sigma + G_1'\left(\Psi_1^2\right).$$

Obviously, Ψ_1 does not depend on σ, and Ψ_1 has the form(110), but with $\sigma = 0$ in the definition of $\mathring{\psi}_1$ in (111). Such a $\mathring{\psi}_1$ satisfies the equation

$$\partial_3^2 \mathring{\psi}_1 - G_{a1}'\left(\mathring{\psi}_1^* \mathring{\psi}_1\right)\mathring{\psi}_1 = 0, \tag{127}$$

and from (126) we obtain that

$$Q = -\frac{\zeta^2}{\Psi}\partial_\tau^2 \Psi + \frac{\zeta^2}{\Psi}\partial_\tau \beta \partial_z \Psi + \frac{2\zeta^2}{\Psi}\beta \partial_\tau \partial_z \Psi + \frac{\zeta^2}{\Psi}\left(\frac{1}{\gamma^2} - \frac{1}{\gamma_0^2}\right)\partial_z^2 \Psi + 2\zeta^2 \sigma. \tag{128}$$

Now we rewrite the complex equation (123) as a system of two real equations. The real part of (123) divided by Ψ yields the following quadratic equation

$$Q - 2\gamma\Phi + \Phi^2 - 2\beta\gamma Z - Z^2 = 0. \tag{129}$$

The small solution Z to this equation is given by the formula

$$Z = \Theta(\tau, \Phi) = -\beta\gamma + \left(\Phi^2 - 2\gamma\Phi + \beta^2\gamma^2 + Q\right)^{1/2}\frac{\beta}{|\beta|}. \tag{130}$$

The imaginary part of (123) divided by $\zeta\Psi$ yields

$$-2\Phi\left(\partial_\tau - \beta\partial_z\right)\ln\Psi - \left(\partial_\tau - \beta\partial_z\right)\Phi + \partial_z Z + 2Z\partial_z \ln\Psi = 0, \tag{131}$$

the coefficients are expressed in terms of the given Ψ defined by (110), (111), and can be written explicitly:

$$\partial_z \ln\Psi = -\gamma_0^2 z, \quad \partial_\tau \ln\Psi = \partial_\tau \sigma, \quad \gamma_0 = \left(1 - \frac{1}{\mathrm{c}^2}v^2(0)\right)^{-1/2}. \tag{132}$$

To determine a *small* solution Φ of (131), (130) in the strip Ξ we impose the condition

$$\Phi = 0 \text{ if } z = 0, \quad -\infty < \tau < \infty. \tag{133}$$

The exact solution Φ of the equation (131), where $Z = \Theta(\tau, \Phi)$ satisfies (130), is a solution of the following quasilinear first-order equation

$$\partial_\tau \Phi - \beta\partial_z \Phi - \Theta_\Phi(\tau, \Phi)\partial_z \Phi = -2\Phi\zeta\left(\partial_\tau - \beta\partial_z\right)\ln\Psi + 2\Theta\partial_z \ln\Psi + \Theta_z, \tag{134}$$

where Θ_Φ and Θ_z are partial derivatives of $\Theta(\tau, \Phi)$ (Θ depends on z via $Q = \operatorname{Re} Q$) with Φ satisfying condition (133). According to the method of characteristics, we write based on (134) the following equations for the characteristics:

$$\frac{d\tau}{ds} = 1, \quad \frac{dz}{ds} = -\Theta_\Phi(\tau, \Phi) - \beta(\tau), \tag{135}$$

$$\frac{d\Phi}{ds} = -2\Phi\left(\partial_\tau - \beta\partial_z\right)\ln\Psi + 2\Theta(\tau, \Phi)\partial_z \ln\Psi + \Theta_z, \tag{136}$$

with the initial data derived from (133) on the line $z = 0$:

$$\tau_{s=0} = \tau_0, \quad z_{s=0} = 0, \quad \Phi_{s=0} = 0. \tag{137}$$

Note that $\Psi = \pi^{-1/4} \mathrm{e}^{-\gamma_0^2 z^2/2+\sigma}$ is a given function of τ, z, and β, γ are given functions of τ. By the method of characteristics, the solution of (134), (133) can be found by integration along the integral curves. If the right-hand side in (136) vanishes at $\Phi = 0$, the exact solution Φ would be zero along the curve for every τ_0; hence it would be zero in Ξ. Therefore the magnitude of Φ on the curve in the strip Ξ is determined by the magnitude of the remaining term,

$$F_1 = 2\Theta(\tau, 0)\, \partial_z \ln \Psi + \Theta_z(\tau, 0), \tag{138}$$

in the right-hand side. The system (135)-(137) is still too complex to hope for a closed form solution, but here we want only to show that this solution is small; this implies, in turn, that the potential φ_2 is small as well. To this end, we show below that (135)-(136) is a small perturbation of a simpler system which corresponds to the principal part of (134) for small ζ.

3.5 Small auxiliary phases and small balancing potential

Now we estimate the leading part of Φ for small ζ. We obtain from (128) the estimate

$$|Q| + |\partial_z Q| + |\partial_\tau Q| = O\left(\epsilon\zeta^2 + \epsilon_1\zeta^2\right). \tag{139}$$

Obviously, if $Q = 0$ then $\Theta(\tau, 0) = 0$. Since in the formula (130) $\beta^2\gamma^2 > 0$, this formula determines for small ζ a smooth function Θ of small $\Phi(\tau, z)$ in the strip $\Xi(\theta)$, and one can see from (128) that the expression for Q is a quadratic polynomial in z with the common factor ζ^2 and bounded coefficients. Therefore $\Theta(\tau, \Phi)$ is a regular function of *small* Φ in the strip $\Xi(\theta)$ as long as

$$\zeta^2\theta^2 + \zeta^2 << 1. \tag{140}$$

Hence we can take

$$\theta \to \infty \quad \text{as} \quad \zeta \to 0, \tag{141}$$

and, therefore, the width θ of the strip where we have the exact solution of (134), (130) is arbitrary large. If $\Phi = O\left(\zeta^2\epsilon + \zeta^2\epsilon_1\right)$ (certainly this assumption is true for small $|z|$ thanks to (133), the argument below shows that we can extend this estimate to the entire strip $\Xi(\theta)$) we have

$$Z = \Theta(\tau, \Phi) = -\frac{1}{\beta}\Phi + O\left(\zeta^2\epsilon + \epsilon_1\zeta^2\right). \tag{142}$$

This implies the following estimate of F_1 in (138)

$$F_1 = O\left(\zeta^2\epsilon + \zeta^2\epsilon_1\right).$$

If we replace Θ by its principal term from (142), we obtain from (135)-(136) a simpler system for the resulting approximation $\mathring{\Phi}$:

$$\frac{d\tau}{ds} = 1, \quad \frac{dz}{ds} = -\beta + \frac{1}{\beta}, \tag{143}$$

$$\frac{d\mathring{\Phi}}{ds} = -2\mathring{\Phi}\left(\partial_\tau - \beta\partial_z\right)\ln\Psi - \frac{2}{\beta}\mathring{\Phi}\partial_z \ln\Psi, \tag{144}$$

A general solution for this system can be written explicitly. Namely, the characteristic curves are given by

$$\tau = \tau_0 + s, \quad z = z(\tau, \tau_0) = \int_{\tau_0}^{\tau} \frac{1}{\beta}\left(1 - \beta^2\right) d\tau, \tag{145}$$

and the value of the phase $\mathring{\Phi}$ is given by the formula

$$\mathring{\Phi}(\tau, z) = \mathring{\Phi}(\tau_0, 0)\, \frac{\Psi^2(\tau_0, 0)}{\Psi^2(\tau, z)}. \tag{146}$$

The integral curves $\tau, z(\tau, \tau_0)$ obviously cover the entire (τ, z)-plane. Since the original problem (135)-(136) is a perturbation of the above system of order $\zeta^2\epsilon + \zeta^2\epsilon_1$, the nice properties of (143), (144) imply that the characteristic equations (135)-(136) and their integral curves also have nice properties, and Φ and $\mathring{\Phi}$ are close one to another. According to (137) we set $\mathring{\Phi}(\tau_0, 0) = 0$, hence $\mathring{\Phi}(\tau, z) = 0$, and the exact solution Φ of (134), (133) in the strip Ξ is of order $\zeta^2\epsilon + \zeta^2\epsilon_1$ for small ζ.

Let us verify now that the balancing potential is small. The solution of (134), (133), obtained by integrating (135)-(137), satisfies in Ξ estimates

$$|\Phi| = O\left(\left(\zeta^2\epsilon + \zeta^2\epsilon_1\right)|z|\right), \quad |\partial_z\Phi| + \left|\partial_z^2\Phi\right| = O\left(\zeta^2\epsilon + \zeta^2\epsilon_1\right). \tag{147}$$

From (119)-(120) we obtain that

$$\varphi_2 = \frac{m\mathrm{c}^2}{q} O\left(\zeta^2\epsilon\,|z| + \zeta^2\epsilon_1\,|z|\right), \tag{148}$$

and obviously the balancing potential vanishes when $\zeta \to 0$.

Let us show now that the constructed potential $\varphi_{\mathrm{ex}} = \varphi_{\mathrm{ac}} + \varphi_2$ satisfies the asymptotic local homogeneity condition (89). Definition (91) takes the form

$$\frac{1}{R_f} = \frac{\max_{|x-r|\le\theta a}\left(|\partial_x^2\varphi_{\mathrm{ex}}|\right)}{\max_t |\partial_x\varphi_{\mathrm{ac}}|} = \frac{\max_{|z|\le\theta}\left(|\partial_z^2\varphi_{\mathrm{ex}}|\right)}{a\max_\tau |\partial_z\varphi_{\mathrm{ac}}|}. \tag{149}$$

To express the magnitude of the gradient of the potential φ_{ac} in terms of the parameters corresponding to the mild accelerations, we use relations (116) and (97):

$$\partial_z\varphi_{\mathrm{ac}} = a\partial_y\varphi_{\mathrm{ac}} = -a\frac{m}{q}\partial_t(\gamma v) = -\frac{m\mathrm{c}^2}{q}\partial_\tau(\gamma\beta),$$

resulting in the estimate

$$\max_\tau |\partial_z\varphi_{\mathrm{ac}}| = \frac{m\mathrm{c}^2}{q} O(\epsilon). \tag{150}$$

Using (119), (120) and taking into account the definition of a_{C}, we obtain

$$\partial_z^2\Phi + \partial_\tau\partial_z Z - \beta\partial_z^2 Z = \frac{q}{m\mathrm{c}^2}\partial_z^2\varphi_2. \tag{151}$$

We observe that, since φ_{ac} is linear, $\partial_z^2\varphi_{\mathrm{ex}} = \partial_z^2\varphi_2$, and using (147) and (142), we conclude that in the strip Ξ

$$\partial_z^2\varphi_{\mathrm{ex}} = \frac{m\mathrm{c}^2}{q} O\left(\zeta^2\epsilon + \epsilon_1\zeta^2\right).$$

Consequently, formula (149) takes the form

$$\frac{1}{R_f} = \frac{O\left(\zeta^2\epsilon + \epsilon_1\zeta^2\right)}{aO\left(\epsilon\right)} = \frac{1}{a}O\left(\zeta^2 + \frac{\epsilon_1}{\epsilon}\zeta^2\right). \tag{152}$$

We obtain that (89) holds in the asymptotic regime

$$\zeta \to 0, \quad \frac{\epsilon_1}{\epsilon}\zeta^2 \to 0, \tag{153}$$

which is consistent with the assumptions (103), (102). The above analysis shows also that the magnitude of the balancing potential φ_2 is vanishingly small compared with the magnitude of the accelerating potential.

The summary of the above argument is that we can find a potential $\varphi_{\rm ex}\left(t,x\right) = \varphi_{\rm ac} + \varphi_2$ such that (i) the function ψ' with the Gaussian profile defined by (109) is an exact solution of the KG equation (107) in a widening strip $\Xi\left(\theta\right)$; (ii) the balancing potential $\varphi_2\left(t,x\right)$ vanishes asymptotically, and the relative magnitude of the balancing potential $\varphi_2\left(t,x\right)$ compared with the accelerating potential $\varphi_{\rm ac}$ tends to zero as $\zeta \to 0$. The charge distribution ψ provides an example of a localized solution in its strongest form, namely, an accelerating solution with a fixed Gaussian shape. The possibility to preserve localization which we discuss in this section concerns microscopic details of the charge evolution, and naturally the ratio $\zeta = a_{\rm C}/a$ of the charge size to the Compton wavelength $a_{\rm C}$ plays an important role. In the context of Section 2.6, a is vanishingly small compared with the macroscopic scale R_f as (152) clearly shows.

Acknowledgment. The research was supported through Dr. A. Nachman of the U.S. Air Force Office of Scientific Research (AFOSR), under grant number FA9550-11-1-0163.

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